VOLUMES WITH KNOWN CROSS SECTIONS: NOTES

To find the area under a curve, we have seen that the Fundamental Theorem of Calculus states:

$$\int_a^b f(x)dx = F(b) - F(a) = Area$$

So, in other words, if you integrate a function with the appropriate antiderivative technique, and then evaluate the difference between the integrated function at its upper limit, $x = b$, and its lower limit, $x = a$, you will solve for the net area between the function and the x-axis.

Taking this concept, we can use the following procedure to find the volume of a solid over a closed interval.

1. Determine what shape the base of the solid will be. In most cases, the base of the solid you are constructing is simply a function or pair of functions with boundaries on the top, bottom, and sides.
2. Determine what shape you want the perpendicular cross sections to be. This will serve as the area function. In most problems, the cross-sectional shape is given.
3. Embed the functions that are serving as the base of the solid into the area equation to create an area function. (Remember that the distance of the cross section will be the upper function minus the lower function.)
4. Evaluate the definite integral you created as if it were no different than a typical definite integral.

Using these steps, let's walk through an example:

The base of a solid is the region bounded by the curve $y = x^2$, the vertical line $x = 2$, and the $x - axis$. Find the volume of the solid if all cross-sections perpendicular to the $x - axis$ are squares.

1. Graph the base of the solid.
2. Draw in a cross-sectional line that is perpendicular to the x-axis. This represents one of an infinite number of cross-sections with area represented by $A = s^2$
3. The line drawn represents the bottom of one of our square cross sections. If you measured the length of s (in this case the distance from the $x - axis$ $(y = 0)$ to the curve $y = x^2$), then the distance s is determined to be $s = x^2 - 0$. One end of the cross section is represented by $y = x^2$ and the other end is represented by $y = 0$.
4. To find the volume, set up the integral:

$$Vol = \int_a^b A(x)dx$$

where $A(x) = (x^2 - 0)^2$, because we are embedding $x^2 - 0$ into the equation for area of a square, $a = 0$ and $b = 2$.

5. The definite integral should now read:

$$Vol = \int_0^2 (x^2 - 0)^2 dx = \int_0^2 x^4 dx = \left[\frac{x^5}{5}\right]_0^2 = \frac{(2)^5}{5} - \frac{(0)^5}{5} = \frac{32}{5} units^3$$