Limits towards Infinity in Rational Functions

Objective: The goal of this exploration is to determine the end behavior of rational function as the values of *x* approach infinity.

Background: You may recall that in arithmetic a quotient is the outcome of one number being divided into another.

For example 2 is the quotient of $\frac{6}{3}$.

When dealing with functions, a quotient function is general term for a function that is comprised of one function divided by another.

For example $f(x) = \frac{\sin(x)}{x}$. In this case f is defined as the sine function, $\sin(x)$, divided by the linear function, x.

A rational function is simply a specific type of quotient function that divides one polynomial by another. We define rational functions as $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

For example, $f(x) = \frac{x^2+6x+1}{x^3-9}$ is a rational function since the numerator and denominator are both polynomials. The numerator is a quadratic polynomial and the denominator is a cubic polynomial.

Exploration:

Situation 1: Let f be a function defined as $f(x) = \frac{1}{x^2}$

A. What is the degree of the polynomial in the numerator? What is the degree of the polynomial in the denominator?

B. Use your calculator to fill in the table below:

x	$f(x)$
1	
10	
50	
100	
500	
1000	
10000	

C. What do notice about the $f(x)$ values as x becomes larger? Are the outputs becoming larger, or smaller?

D. If you feel like the values are getting larger, where is the function diverging towards? If you feel like the values are becoming smaller, where do you think the values are converging towards?

E. Determine the limit, $\lim_{x\to\infty}\frac{1}{x^2} =$ ________ . Where do you think the function will tend towards as x approaches negative infinity? Use your calculator to find $\lim_{x\to-\infty}\frac{1}{x^2}$

Situation 2: Let g be a function defined as $g(x) = \frac{x^3-1}{x+5}$

A. What is the degree of the polynomial in the numerator? What is the degree of the polynomial in the denominator?

B. Use your calculator to fill in the table below:

x	$g(x)$
1	
10	
50	
100	
500	
1000	
10000	

C. What do notice about the $f(x)$ values as x becomes larger? Are the outputs becoming larger, or smaller?

D. If you feel like the values are getting larger, where is the function diverging towards? If you feel like the values are becoming smaller, where do you think the values are converging towards?

E. How is this outcome different than situation one?

F. Do you notice any differences in the degree of the polynomials in the numerator and denominator?

G. Determine the limit, $\lim_{x \to \infty} \frac{x^3-1}{x+5} =$ ________ . Where do you think the function will tend towards as x approaches negative infinity? Use your calculator to find $\lim_{x \to -\infty} \frac{x^3-1}{x+5}$

Situation 3: Let h be defined as the function $h(x) = \frac{3x^2-6x+1}{4-2x^2}$

A. What is the degree of the polynomial in the numerator? What is the degree of the polynomial in the denominator?

B. Use your calculator to fill in the table below:

x	$g(x)$
1	
10	
50	
100	
500	
1000	
10000	

C. What do notice about the $f(x)$ values as x becomes larger? Are the outputs becoming larger, or smaller?

D. If you feel like the values are getting larger, where is the function diverging towards? If you feel like the values are becoming smaller, where do you think the values are converging towards?

E. How is this outcome different than situations one and two? Are there similarities?

F. Do you notice any differences in the degrees of the polynomials in the numerator and denominator?

G. What connections can you make between your solution and the coefficients that correspond to the highest powers of x in the numerator and denominator?

H. Determine the limit, $\lim_{x\to\infty} \frac{3x^2-6x+1}{4-2x^2} =$ ________ . Where do you think the function will tend towards as x approaches negative infinity? Use your calculator to find $\lim_{x\to-\infty} \frac{3x^2-6x+1}{4-2x^2}$

Follow up:

1. What generalizations can you make about limits in rational functions when the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator?

2. What generalizations can you make about limits in rational functions when the degree of the polynomial in the numerator is greater than the degree of the polynomial in the denominator?

3. What about when the degree in the numerator and the denominator are the same?

4. Do you think this is true for all rational functions? Explain.

5. In your own words, explain how limits towards infinity in rational functions and the end behavior of rational functions are similar?